THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 1
(Prepared by Tai Ho Man)

0.1 Complex Numbers and Basic Algebraic Manipulations

We usually denote $\mathbb{C} = \{a + bi : a \text{ and } b \text{ are real numbers}\} =$ All complex numbers.

Definition 1. *If $z \in \mathbb{C}$ and $z = a + bi$, then this complex number is expressed in* ***standard form****. And a is called the* ***real part*** *of z and is denoted by Re(z). Similarly, b is called the* ***imaginary part*** *of z and is denoted by Im(z). We say z is* ***real*** *if $b = 0$ and z is* ***purely imaginary*** *if $a = 0$.*

Remark: Be careful that both Re(z) and Im(z) are real numbers !!

Proposition 1. *(Algebraic Operation) Suppose that z, w and u are complex numbers, then the followings hold:*

- *Commutative Law (addition): $z + w = w + z$;*
- *Associative Law (addition): $z + (w + u) = (z + w) + u$;*
- *Commutative Law (multiplication): $zw = wz$;*
- *Associative Law (multiplication): $z(wu) = (zw)u$;*
- *Distributive Law : $z(w + u) = zw + zu$.*

Remark: Actually the operation of addition and multiplication is the same as the one of real number. The above properties can be proved by direct computation.

Definition 2. *For any complex number $z = a + bi$, we define the* ***conjugate*** *of z to $a - bi$ and is denoted by $\overline{z}$.*

Remark: It is just a reflection along x-axis (real axis).

Theorem 1. *(Euler's Formulae) For any complex number $z = a + bi$, we have*

$$e^z = e^{a+bi} = e^a e^{bi} = e^a(\cos b + i \sin b)$$

Remark: It can be easily proved by Taylor expansion of e^x.

From the powerful Eulers Formulae, we are motivated to consider the polar form of complex number.

Proposition 2. *If we write $z = a + bi$, the* ***polar form*** *of z is given by $z = |z|e^{i\theta}$, where $|z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. We called $|z|$ the* ***modulus*** *or* ***length*** *of z. We called θ the* ***argument*** *of z.*

Remark 1 : If θ_0 is the argument of z, then $\theta_0 + 2\pi$ is also the argument of z due to the periodicity of tan function. To avoid writing multi-value of argument of z, if we restrict argument θ in $(-\pi, \pi]$, we will called this unique value to be the **principal argument** of z and denote it by Arg(z). And the argument of z is defined by the set :
arg(z)={ Arg(z)+$2k\pi$: k is an integer}.

Remark 2 : We can easily obtain the fact that $|zw| = |z||w|$ and arg(zw) =arg(z)+arg(w). But it is not true that Arg(zw) =Arg(z)+Arg(w). (try the example $z = i$ and $w = -1$)

Theorem 2. *(Triangle inequality) Suppose that z_1 , ..., z_n are complex numbers, then it holds*

$$|z_1 + ... + z_n| \leq |z_1| + .. + |z_n|.$$

0.2 Some Geometric Objects Represented In Complex Theory

The parametric representation of a straight line passing through the point a and having the direction v is given by $z = a + tv$ where t is a real number.

The parametric representation of a straight line passing through the point a and b is given $z = (1-t)a + tb$ where t is a real number. If we only consider the line segment from a to b, the parametric representation of this line segment is given by $z = (1-t)a + tb$ with $t \in [0, 1]$.

If the straight line is a perpendicular bisector of the line segment joining a and b, then the set representation of this straight line is given by $\{z \in \mathbb{C} : |z - a| = |z - b|\}$.

The set representation of the straight line passing through the point a and b is given by $\left\{z \in \mathbb{C} : Im\left(\frac{z-a}{a-b} = 0\right)\right\}$.

The parametric representation of a circle centred at the point a and having radius $r > 0$ is given by $z = a + re^{i\theta}$ where $\theta \in [0, 2\pi)$.

The set representation of a circle centred at the point a and having radius $r > 0$ is given by $\{z \in \mathbb{C} : |z - a| = r\}$.

The set representation of a circle passing through the point a, b and c clockwisely is given by $\left\{z \in \mathbb{C} : Im\left(\frac{z-a}{z-b} \cdot \frac{c-b}{c-a} = 0\right)\right\}$.

Proposition 3. *(The roots of unity) If $z^n = a = re^{i\theta}$ with $\theta \in (-\pi, \pi]$, then $z = r^{1/n}e^{i(\frac{\theta+2k\pi}{n})}$ for $k = 0, 1, 2, ..., n-1$.*

Definition 3. *Let $\Omega \subset \mathbb{C}$, Ω is an open set if for any $z_0 \in \Omega$ there is a $r > 0$ such that $B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\} \subset \Omega$.*

Definition 4. *Let $T \subset \mathbb{C}$, T is a closed set if its complement set $\mathbb{C} \setminus T$ is open.*

0.3 Exercise :

1. Prove that $|zw| = |z||w|$ and arg(zw) =arg(z)+arg(w). Also give a example to disprove Arg(zw) =Arg(z)+Arg(w)

2. Show that Re $(z) \leq |z|$ and Im$(z) \leq |z|$.

3. Express $(1 + i)^5$ in standard form.

4. Prove the inverse triangle inequality: $\big||z| - |w|\big| \leq |z - w|$.